

# Algebraic Bethe ansatz for open XXX model with triangular boundary matrices

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## Abstract

*We consider open XXX spins chain with two general boundary matrices submitted to one constraint, which is equivalent to the possibility to put the two matrices in a triangular form. We construct Bethe vectors from a generalized algebraic Bethe ansatz. As usual, the method also provides Bethe equations and transfer matrix eigenvalues.*

XXX spin chain [1] is one of the most studied integrable model. When the boundary conditions are the periodic ones, almost everything is known about it, starting from its spectrum up to asymptotics of correlation functions, and all the possible methods (e.g. coordinate [2] or algebraic Bethe ansatz [3, 4, 5, 6],...) have been successfully used. When the model has open boundary conditions, the situation changes drastically. Indeed, although the model is known to be integrable for general boundary matrices, for a long time only the case of diagonal boundary matrices was well-understood [7]. Recently [8, 9], a first step toward full resolution of the model was done, in the framework of algebraic or coordinate Bethe ansatz, and when one boundary matrix is triangular. Let us also mention [10] where the separation of variables has been used to tackle this problem. The aim of the present letter is to construct the algebraic Bethe ansatz for open XXX spin chain with two general boundary matrices with one relation among their entries. This relation is equivalent to the possibility to put the two matrices in a triangular form.

When the two boundaries are triangular, the main difficulty lies in the fact that the number of excitations is not preserved anymore. To overcome this problem, we allow the ansatz to have states with unfixed number of excitations. The same idea was applied successfully in [9, 11, 12] in the context of coordinate Bethe ansatz.

## 1 Reflection equation and transfer matrix

The open XXX spin chain Hamiltonian can be constructed from the rational  $R$ -matrix, satisfying the Yang-Baxter equation [13, 14], and  $K$ -matrices, satisfying the reflection equation [15, 7]. In this section, we will recall the main step of this construction and give the constraints on the parameters to get triangular boundary matrices. We also present its pseudo-vacuum state.

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## 1.1 Reflection equation

The R-matrix associated to the XXX spin chain has the well-known form

$$R(u) = \begin{pmatrix} \mathfrak{a}(u) & & & \\ & \mathfrak{b}(u) & \mathfrak{c} & \\ & \mathfrak{c} & \mathfrak{b}(u) & \\ & & & \mathfrak{a}(u) \end{pmatrix} \quad (1)$$

where

$$\mathfrak{a}(u) = u + \eta \quad ; \quad \mathfrak{b}(u) = u \quad \text{and} \quad \mathfrak{c} = \eta . \quad (2)$$

This R-matrix, associated to the Yangian  $Y(\mathfrak{sl}_2)$ , satisfies the Yang-Baxter equation [13, 14]

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v) \quad (3)$$

where, as usual, the subscripts of the R-matrix indicate the spaces where the R-matrix acts non trivially. This R-matrix allows one to construct the monodromy matrix

$$T_a(u) = R_{a1}(u - \xi_1) R_{a2}(u - \xi_2) \dots R_{aL}(u - \xi_L) , \quad (4)$$

where  $L$  is the number of sites and  $\xi_j$  are free parameters, called inhomogeneity parameters. This monodromy matrix is the cornerstone of the study of integrable periodic spin chains.

To construct integrable spin chains with boundaries, we follow the method introduced in [15, 7] based on the reflection equations

$$R_{12}(u-v) B_1(u) R_{12}(u+v) B_2(u) = B_2(u) R_{12}(u+v) B_1(u) R_{12}(u-v) , \quad (5)$$

$$R_{12}(-u+v) \overline{B}_1^t(u) R_{12}(-u-v-2\eta) \overline{B}_2^t(v) = \overline{B}_2^t(v) R_{12}(-u-v-2\eta) \overline{B}_1^t(u) R_{12}(-u+v) , \quad (6)$$

where  $.^t$  stands for the transposition. For later convenience, we introduce the following operators

$$\mathcal{A}(u) = B_{11}(u) , \quad \mathcal{B}(u) = B_{12}(u) , \quad \mathcal{C}(u) = B_{21}(u) , \quad \mathcal{D}(u) = B_{22}(u) - \frac{\eta}{2u+\eta} B_{11}(u) , \quad (7)$$

where  $B_{ij}(u)$  are the entries of the matrix  $B(u)$ . The reflection equation (5) provides commutation relations between these operators. This computation is well-known [7] (see also [16]) and we report a list of these commutation relations in Appendix A.

To construct an open spin chain, we need scalar solutions of these reflection equations. In the  $Y(\mathfrak{sl}_2)$  case, the most general scalar solutions are well-known and given, respectively for (5) and (6), by

$$K(u) = \begin{pmatrix} u\beta + \alpha & u\gamma \\ u\delta & -u\beta + \alpha \end{pmatrix} \quad \text{and} \quad \overline{K}(u) = \begin{pmatrix} (-u-\eta)\bar{\beta} + \bar{\alpha} & (-u-\eta)\bar{\gamma} \\ (-u-\eta)\bar{\delta} & (u+\eta)\bar{\beta} + \bar{\alpha} \end{pmatrix} , \quad (8)$$

where  $\alpha, \beta, \gamma, \delta$  and  $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$  are free parameters. Using the monodromy matrix  $T(u)$  and the scalar solution  $K(u)$ , we construct another solution of the reflection equation (5) via the dressing procedure

$$B_a(u) = T_a(u) K_a(u) T_a^{-1}(-u) . \quad (9)$$

We are now in position to introduce the transfer matrix associated to open XXX spin chain

$$t(u) = \text{tr}_a \left( \overline{K}_a(u) B_a(u) \right) , \quad (10)$$

which commutes for different spectral parameters (i.e.  $[t(u), t(v)] = 0$ ). Finally, the integrable open XXX spin chain Hamiltonian is given by (in the case where  $\xi_j = 0$ )

$$H_{XXX \text{ open}} = \frac{\eta^{2L-1}}{8\alpha\bar{\alpha}} \frac{d}{du} t(u)|_{u=0} = \sum_{j=1}^{L-1} P_{j,j+1} + \frac{1}{2\bar{\alpha}} \overline{K}_1(0) + \frac{1}{4\eta\alpha} K'_L(0). \quad (11)$$

## 1.2 Triangularization and constraints on the boundary parameters

Although the spin chain with boundaries provided by the  $K$ -matrices (8) is integrable (i.e. there exist  $L$  conserved charges), the computation of its eigenvectors remains an open problem. The main difficulty lies in the construction of a pseudo-vacuum, i.e. the determination of one particular Hamiltonian eigenvector. The case with diagonal boundaries does not share this difficulty and has been already treated in [7]. Then, using the  $R$ -matrix invariance, the case where  $K$  and  $\bar{K}$  can be simultaneously diagonalized has been treated in [17, 18]. The most general case treated up to now is when there exists a basis where  $K$  is triangular and  $\bar{K}$  is diagonal [8]. In this paper, we propose a generalization of the algebraic Bethe ansatz (based on the ideas of papers [11, 9, 12]) to deal with the case when both  $K$ -matrices are triangular.

Obviously [17, 18, 8], if we conjugate both  $K$ -matrices by a constant matrix, the transfer matrix eigenvalues are unchanged. Then, we want to find a 2 by 2 matrix  $M$  such as  $M^{-1}K(u)M$  and  $M^{-1}\bar{K}(u)M$  are upper triangular. Unfortunately, it is not always possible. It is a simple algebra exercise to show that one can do it if and only if the following constraint is valid

$$(\bar{\delta}\gamma - \delta\bar{\gamma})^2 - 4(\beta\bar{\gamma} - \bar{\beta}\gamma)(\bar{\delta}\beta - \delta\bar{\beta}) = 0. \quad (12)$$

To our knowledge, it is the less restrictive constraint on the boundary parameters for which the eigenvalues and the eigenvectors of the transfer matrix are known (see Section 2.2). Evidently, we can deal with lower triangular matrices in the same way.

## 1.3 Pseudo-vacuum

From now on, we assume that the constraint (12) is satisfied. Then, we can triangularize the  $K$ -matrices (8) to get the following  $K$ -matrices

$$K(u) = \begin{pmatrix} u b + a & u c \\ 0 & -u b + a \end{pmatrix} \quad \text{and} \quad \bar{K}(u) = \begin{pmatrix} -(u + \eta) \bar{b} + \bar{a} & -(u + \eta) \bar{c} \\ 0 & (u + \eta) \bar{b} + \bar{a} \end{pmatrix}, \quad (13)$$

where  $a, b, c$  and  $\bar{a}, \bar{b}, \bar{c}$  are still free parameters. The relations between  $a, b, c, \bar{a}, \bar{b}$  and  $\bar{c}$  and the original parameters are given by

$$a = \alpha, \quad b^2 = \beta^2 + \gamma\delta, \quad c = \gamma - (\beta + b) \quad (14)$$

and similar relations with “bar” parameters.

The transfer matrix to be diagonalized now reads

$$t(u) = \kappa_1(u) \mathcal{A}(u) + \kappa_2(u) \mathcal{D}(u) + \kappa_{12}(u) \mathcal{C}(u), \quad (15)$$

where we used the notations

$$\kappa_1(u) = \frac{2(u + \eta)}{2u + \eta} (\bar{a} - \bar{b}u) \quad , \quad \kappa_2(u) = (u + \eta) \bar{b} + \bar{a} \quad , \quad \kappa_{12}(u) = -(u + \eta) \bar{c}. \quad (16)$$

An important point is that for two triangular matrices, there still exists a simple eigenvector, called pseudo-vacuum. Indeed, let us consider the vector  $|\Omega\rangle$  with  $L$  spin up i.e.

$$|\Omega\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (17)$$

As explained in [8], when we choose  $K$ -matrices (13), this vector has the following properties

$$\mathcal{A}(u)|\Omega\rangle = \Lambda_1(u)|\Omega\rangle \quad \text{with} \quad \Lambda_1(u) = (a + bu) \prod_{j=1}^L \frac{\mathbf{a}(u - \xi_j)}{\mathbf{a}(-u - \xi_j)}; \quad (18)$$

$$\mathcal{D}(u)|\Omega\rangle = \Lambda_2(u)|\Omega\rangle \quad \text{with} \quad \Lambda_2(u) = \frac{2u(a - b(u + \eta))}{2u + \eta} \prod_{j=1}^L \frac{\mathbf{b}(u + \xi_j)\mathbf{b}(u - \xi_j)}{\mathbf{a}(u + \xi_j)\mathbf{a}(-u - \xi_j)}; \quad (19)$$

$$\mathcal{C}(u)|\Omega\rangle = 0. \quad (20)$$

Using these properties, it is easy to show that  $|\Omega\rangle$  is an eigenvector of  $t(u)$

$$t(u)|\Omega\rangle = (\kappa_1(u)\Lambda_1(u) + \kappa_2(u)\Lambda_2(u)) |\Omega\rangle. \quad (21)$$

## 2 Algebraic Bethe Ansatz

### 2.1 Bethe vectors

We are now in position to propose an ansatz for all the eigenvectors. Let us remark that if  $\bar{c}$  vanishes, the ansatz used to study the case with both boundaries diagonal is still working. This trick was used previously in [8]. For the case with  $\bar{c}$  different from 0, we need to generalize the ansatz. To this aim, we borrow the idea from the papers [11, 9, 12] where, instead of having a fixed number of excitations, we only fix  $N$ , the maximum of number of excitations.

Before giving the explicit form of the Bethe eigenvectors, we need some definitions:

$$\mathcal{I}_\ell^N = \{ \{i_1, i_2, \dots, i_\ell\} \mid 1 \leq i_1 < \dots < i_\ell \leq N \}, \quad 0 \leq \ell \leq N \quad (22)$$

$$\mathcal{I}^N = \bigcup_{\ell=0}^N \mathcal{I}_\ell^N, \quad (23)$$

$$\bar{I} = \{1, 2, \dots, N\} \setminus I, \quad \forall I \in \mathcal{I}^N. \quad (24)$$

Starting from a set of  $N$  elements  $\mathbf{u} = \{u_1, \dots, u_N\}$ , we define  $\mathbf{u}_I = \{u_i \mid i \in I\}$  for any  $I \in \mathcal{I}^N$ .

We can now define the Bethe vectors which depend on the Bethe parameters  $\mathbf{u} = \{u_1, u_2, \dots, u_N\}$ :

$$|\Phi^N(\mathbf{u})\rangle = \sum_{I \in \mathcal{I}^N} W_I(\mathbf{u}) |\mathbb{B}(\mathbf{u}_I)\rangle. \quad (25)$$

We have used the following vectors:

$$|\mathbb{B}(\mathbf{u}_I)\rangle = \prod_{i \in I} \mathcal{B}(u_i) |\Omega\rangle, \quad I \in \mathcal{I}^N \quad (26)$$

and the following functions

$$W_I(\mathbf{u}) = \frac{\prod_{i \in \bar{I}} \left( \Lambda_2(u_i) \frac{\bar{c}(2u_i + \eta)}{2(\bar{b}u_i - \bar{a})} \prod_{\substack{k=1 \\ k \neq i}}^N h(u_i, u_k) \right)}{\prod_{\substack{j, k \in \bar{I} \\ j < k}} h(u_j, u_k) f(u_j, u_k)}, \quad I \in \mathcal{I}^N, \quad (27)$$

where the functions  $f$  and  $h$  are defined in (59).

Before giving the main result of this paper, let us comment the form of the Bethe vectors to clarify notations and conventions. We use the usual convention that a product over an empty set is equal to one, e.g. if  $I \in \mathcal{I}_N^N$ , we get  $\bar{I} = \emptyset$  and  $W_I = 1$ . We deduce that, for  $\bar{c} = 0$ , we get

$$|\Phi^N(\mathbf{u})\rangle = \prod_{i=1}^N \mathcal{B}(u_i) |\Omega\rangle, \quad (28)$$

that corresponds to the usual ansatz, as expected, since for  $\bar{c} = 0$  the left boundary becomes diagonal and the operator  $\mathcal{C}$  is not present anymore in the transfer matrix (15). For  $N = 0$ , the

Bethe vector  $|\Phi^0\rangle$  reduces to the pseudo-vacuum  $|\Omega\rangle$ . Finally, let us give more explicitly the Bethe vectors for the cases  $N = 1, 2$ :

$$|\Phi^1(u_1)\rangle = \left( \mathcal{B}(u_1) + \Lambda_2(u_1) \frac{\bar{c}(2u_1 + \eta)}{2(\bar{b}u_1 - \bar{a})} \right) |\Omega\rangle \quad (29)$$

$$|\Phi^2(u_1, u_2)\rangle = \left( \mathcal{B}(u_1)\mathcal{B}(u_2) + W_{\{2\}}(u_1, u_2)\mathcal{B}(u_2) + W_{\{1\}}(u_1, u_2)\mathcal{B}(u_1) + W_\emptyset(u_1, u_2) \right) |\Omega\rangle \quad (30)$$

with

$$W_{\{2\}}(u_1, u_2) = \Lambda_2(u_1) \frac{\bar{c}(2u_1 + \eta)}{2(\bar{b}u_1 - \bar{a})} \frac{(u_1 - u_2 + \eta)(u_1 + u_2 + 2\eta)}{(u_1 - u_2)(u_1 + u_2 + \eta)}, \quad (31)$$

$$W_\emptyset(u_1, u_2) = \Lambda_2(u_1) \Lambda_2(u_2) \frac{\bar{c}(2u_1 + \eta)}{2(\bar{b}u_1 - \bar{a})} \frac{\bar{c}(2u_2 + \eta)}{2(\bar{b}u_2 - \bar{a})} \frac{(u_1 + u_2 + 2\eta)}{(u_1 + u_2)}. \quad (32)$$

## 2.2 Eigenvectors and eigenvalues

We arrive at the main results of this paper. The Bethe vector (25) is an eigenvector of the transfer matrix  $t(u)$  defined by (15) i.e. we get

$$t(u)|\Phi^N(\mathbf{u})\rangle = \Lambda(u)|\Phi^N(\mathbf{u})\rangle \quad (33)$$

with

$$\Lambda(u) = \kappa_1(u)\Lambda_1(u) \prod_{k=1}^N f(u, u_k) + \kappa_2(u)\Lambda_2(u) \prod_{k=1}^N h(u, u_k) \quad (34)$$

if the Bethe parameters  $\{u_1, u_2, \dots, u_N\}$  satisfy the following Bethe equations

$$\frac{\Lambda_1(u_k)}{\Lambda_2(u_k)} = \Xi(u_k) \prod_{j \neq k}^N \frac{h(u_k, u_j)}{f(u_k, u_j)} \quad (35)$$

with

$$\Xi(u) = \frac{(2u + \eta)(\bar{b}(u + \eta) + \bar{a})}{2u(\bar{a} - \bar{b}u)}. \quad (36)$$

The proof of this result is given in Section 3.

## 3 Proof of relation (33)

### 3.1 Actions of $\mathcal{A}(u)$ , $\mathcal{D}(u)$ and $\mathcal{C}(u)$

We need the actions of  $\mathcal{A}(u)$ ,  $\mathcal{D}(u)$  and  $\mathcal{C}(u)$  on vectors of type  $|\mathbb{B}(\mathbf{x})\rangle = \prod_{i=1}^\ell \mathcal{B}(x_i)|\Omega\rangle$  where  $\mathbf{x} = \{x_1, \dots, x_\ell\}$  can be any subset of Bethe parameters. The computation for the first two actions is a usual computation [7] using the commutation relations of Appendix A. We get

$$\mathcal{A}(u)|\mathbb{B}(\mathbf{x})\rangle = \Lambda_1^\ell(u, \mathbf{x})|\mathbb{B}(\mathbf{x})\rangle + \sum_{k=1}^\ell M_k(u, \mathbf{x}) \mathcal{B}(u) \prod_{\substack{i=1 \\ i \neq k}}^\ell \mathcal{B}(x_i)|\Omega\rangle \quad (37)$$

$$\mathcal{D}(u)|\mathbb{B}(\mathbf{x})\rangle = \Lambda_2^\ell(u, \mathbf{x})|\mathbb{B}(\mathbf{x})\rangle + \sum_{k=1}^\ell N_k(u, \mathbf{x}) \mathcal{B}(u) \prod_{\substack{i=1 \\ i \neq k}}^\ell \mathcal{B}(x_i)|\Omega\rangle \quad (38)$$

where

$$\Lambda_1^\ell(u, \mathbf{x}) = \Lambda_1(u) \prod_{k=1}^{\ell} f(u, x_k) \quad , \quad \Lambda_2^\ell(u, \mathbf{x}) = \Lambda_2(u) \prod_{k=1}^{\ell} h(u, x_k) \quad (39)$$

and, using the notation  $\mathbf{x}_{\neq k} = \{x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_\ell\}$

$$M_k(u, \mathbf{x}) = g(u, x_k) \Lambda_1^{\ell-1}(x_k, \mathbf{x}_{\neq k}) + w(u, x_k) \Lambda_2^{\ell-1}(x_k, \mathbf{x}_{\neq k}) \quad (40)$$

$$N_k(u, \mathbf{x}) = k(u, x_k) \Lambda_2^{\ell-1}(x_k, \mathbf{x}_{\neq k}) + n(u, x_k) \Lambda_1^{\ell-1}(x_k, \mathbf{x}_{\neq k}). \quad (41)$$

The computation of the action of  $\mathcal{C}(u)$  on  $|\mathbb{B}(\mathbf{x})\rangle$  is more involved, but after some algebra, using commutation relations of Appendix A, we get

$$\mathcal{C}(u)|\mathbb{B}(\mathbf{x})\rangle = \sum_{i=1}^{\ell} G_i(u, \mathbf{x}) |\mathbb{B}(\mathbf{x}_{\neq i})\rangle + \sum_{i < j}^{\ell} F_{ij}(u, \mathbf{x}) \mathcal{B}(u) |\mathbb{B}(\mathbf{x}_{\neq i, j})\rangle. \quad (42)$$

The functions  $F$  and  $G$  are given in terms of the functions defined in (60) and by the following relations:

$$\begin{aligned} G_i(u, \mathbf{x}) &= \Lambda_1^{\ell-1}(u, \mathbf{x}_{\neq i}) \left( (m(u, x_i) + l(u, x_i)) \Lambda_1^{\ell-1}(x_i, \mathbf{x}_{\neq i}) + p(u, x_i) \Lambda_2^{\ell-1}(x_i, \mathbf{x}_{\neq i}) \right) \\ &+ \Lambda_2^{\ell-1}(u, \mathbf{x}_{\neq i}) \left( (q(u, x_i) + y(u, x_i)) \Lambda_1^{\ell-1}(x_i, \mathbf{x}_{\neq i}) + z(u, x_i) \Lambda_2^{\ell-1}(x_i, \mathbf{x}_{\neq i}) \right) \end{aligned} \quad (43)$$

$$\begin{aligned} F_{ij}(u, \mathbf{x}) &= \Lambda_1^{\ell-2}(x_i, \mathbf{x}_{\neq i, j}) (Z_{11}(u, x_i, x_j) \Lambda_1^{\ell-2}(x_j, \mathbf{x}_{\neq i, j}) + Z_{12}(u, x_i, x_j) \Lambda_2^{\ell-2}(x_j, \mathbf{x}_{\neq i, j})) \\ &+ \Lambda_2^{\ell-2}(x_i, \mathbf{x}_{\neq i, j}) (Z_{12}(u, x_j, x_i) \Lambda_1^{\ell-2}(x_j, \mathbf{x}_{\neq i, j}) + Z_{22}(u, x_i, x_j) \Lambda_2^{\ell-2}(x_j, \mathbf{x}_{\neq i, j})) \end{aligned} \quad (44)$$

where

$$Z_{11}(u, x_i, x_j) = \frac{8\eta^2 x_i x_j (x_i + x_j)(u^2 - x_i x_j + \eta u)}{(2x_i + \eta)(2x_j + \eta)(x_i + x_j + \eta)(u + x_i + \eta)(u + x_j + \eta)(u - x_i)(u - x_j)}, \quad (45)$$

$$Z_{12}(u, x_i, x_j) = \frac{4\eta^2 x_i (x_j - x_i + \eta)(u^2 + \eta u + x_i x_j + \eta x_i)}{(2x_i + \eta)(x_i - x_j)(u + x_i + \eta)(u + x_j + \eta)(u - x_i)(u - x_j)}, \quad (46)$$

$$Z_{22}(u, x_i, x_j) = \frac{2\eta^2 (x_i + x_j + 2\eta)(u^2 - (x_i + \eta)(x_j + \eta) + \eta u)}{(x_i + x_j + \eta)(u + x_i + \eta)(u + x_j + \eta)(u - x_i)(u - x_j)}. \quad (47)$$

We want to prove relation (33) using relations (37), (38) and (42). We project the L.H.S and the R.H.S of (33) on all vectors  $|\mathbb{B}(\mathbf{u}_I)\rangle$ ,  $I \in \mathcal{I}^N$ , and all vectors  $\mathcal{B}(u)|\mathbb{B}(\mathbf{u}_I)\rangle$ ,  $I \in \mathcal{I}^N$ , and prove that each relation we get in this way is true.

### 3.2 Projection on $|\mathbb{B}(\mathbf{u})\rangle$

This computation is similar to the usual one (with diagonal boundaries). Thus, we just sketch the proof.

The only contributions of the L.H.S. of (33) are the first terms of relations (37) and (38) for  $\ell = N$ . Using the explicit expressions (39), we recognize the R.H.S. of (33). This proves that the projection of (33) on  $|\mathbb{B}(\mathbf{u})\rangle$  is satisfied.

### 3.3 Projection on $\mathcal{B}(u)|\mathbb{B}(\mathbf{u}_{\neq k})\rangle$

This computation is also similar to the usual one with diagonal boundaries. The only contributions of the L.H.S. of (33) are the second terms of the relations (37) and (38) (for  $\ell = N$ ). There is no contribution from the R.H.S. of (33). Then, the relation to prove is

$$\kappa_1(u) M_k(u, \mathbf{u}) + \kappa_2(u) N_k(u, \mathbf{u}) = 0. \quad (48)$$

Using the explicit forms (40) and (41), the previous equation (48) is equivalent to the Bethe equations (35). This proves the projection on  $\mathcal{B}(u)|\mathbb{B}(\mathbf{u}_I)\rangle$  for  $I \in \mathcal{I}_{N-1}^N$ .

### 3.4 Projection on $|\mathbb{B}(\mathbf{u}_I)\rangle$ for $I \in \mathcal{I}_\ell^N$ and $\ell = 0, 1, \dots, N-1$

This type of projection is not present in the usual diagonal boundary case. Therefore, we give here more details. There are contributions of the L.H.S. of (33) coming from the first terms of relations (37) and (38), from the first term of (42), and contributions of the R.H.S. of (33). The relation to prove is then

$$W_I(\mathbf{u}) \left( \kappa_1(u) \Lambda_1^\ell(u, \mathbf{u}_I) + \kappa_2(u) \Lambda_2^\ell(u, \mathbf{u}_I) - \Lambda(u) \right) + \kappa_{12}(u) \sum_{j \in \bar{I}} W_{I \cup j}(\mathbf{u}) G_j(u, \mathbf{u}_{I \cup j}) = 0. \quad (49)$$

In fact we are going to prove a stronger statement: the coefficient of  $\Lambda_1^\ell(u, \mathbf{u}_I)$  in (49), as well as the coefficient of  $\Lambda_2^\ell(u, \mathbf{u}_I)$  in the same equation, both vanish identically. Indeed, the first coefficient reads

$$\begin{aligned} \text{coef}_1(u) &= \kappa_1(u) W_I(\mathbf{u}) \left( 1 - \prod_{j \in \bar{I}} f(u, u_j) \right) \\ &\quad + \kappa_{12}(u) \sum_{j \in \bar{I}} W_{I \cup j}(\mathbf{u}) \left( (m(u, u_j) + l(u, u_j)) \Lambda_1^\ell(u_j, \mathbf{u}_I) + p(u, u_j) \Lambda_2^\ell(u_j, \mathbf{u}_I) \right) \end{aligned} \quad (50)$$

while the second one is just  $\text{coef}_2(u) = \text{coef}_1(-u - \eta)$ . Thus, if one coefficient vanishes for all  $u$ , so does the other one.

To show that  $\text{coef}_1(u)$  vanishes, we follow the technics used in [11, 9]: we prove that  $\text{coef}_1(u)$  corresponds to the sum over all residues of some function. The function to consider is

$$F(z, u) = \frac{\bar{b}z - \bar{a}}{2\eta z(z - u)} \prod_{j \in \bar{I}} f(z, u_j). \quad (51)$$

The poles of  $F$  (considered as a function of  $z$ ) are located at  $z = u_j, -u_j - \eta, 0, u$  and  $\infty$ . It is a simple exercise to show that the sum over all the residues is then just equivalent to  $\text{coef}_1(u) = 0$  when one replaces the functions  $W_I, \Lambda_k^\ell, \kappa_1, \kappa_{12}, m, l$  and  $p$  by their explicit expression (27), (39), (16) and (60), and uses the Bethe equations (35). This proves the projection on  $|\mathbb{B}(\mathbf{u}_I)\rangle$  for  $I \in \mathcal{I}_\ell^N$  and  $\ell = 0, 1, \dots, N-1$ .

### 3.5 Projection on $\mathcal{B}(u)|\mathbb{B}(\mathbf{u}_I)\rangle$ for $I \in \mathcal{I}_\ell^N$ and $\ell = 0, 1, \dots, N-2$

This type of projection is also new in comparison to the case with diagonal boundaries. Therefore, we give also some details. There are contributions of the L.H.S. of (33) coming from the second terms of relations (37) and (38), from the second term of (42) and no contribution from the R.H.S. of (33). The relation to prove is then

$$\sum_{j \in \bar{I}} W_{I \cup j}(\mathbf{u}) \left( \kappa_1(u) M_j(u, \mathbf{u}_{I \cup j}) + \kappa_2(u) N_j(u, \mathbf{u}_{I \cup j}) \right) + \kappa_{12}(u) \sum_{\substack{j, k \in \bar{I} \\ j < k}} W_{I \cup \{j, k\}}(\mathbf{u}) F_{ij}(u, \mathbf{u}_{I \cup \{j, k\}}) = 0 \quad (52)$$

The previous relation becomes the following functional relation when we used the explicit expressions of  $W_I, M_j, N_j, F_{ij}, \kappa_1, \kappa_{12}$  (see relations (27), (40), (41), (44) and (16)) and the Bethe equations (35)

$$\begin{aligned} 0 &= \sum_{j \in \bar{I}} L(u, u_j) \left( \prod_{\substack{k \in \bar{I} \\ k \neq j}} h_{jk} - \prod_{\substack{k \in \bar{I} \\ k \neq j}} f_{jk} \right) + \frac{1}{2} \sum_{\substack{j, \ell \in \bar{I} \\ j \neq \ell}} \left( Q(u, -u_j - \eta, -u_\ell - \eta) \prod_{\substack{k \in \bar{I} \\ k \neq j, \ell}} f_{jk} f_{\ell k} + \right. \\ &\quad \left. + Q(u, u_j, u_\ell) \prod_{\substack{k \in \bar{I} \\ k \neq j, \ell}} h_{jk} h_{\ell k} + Q(u, -u_j - \eta, u_\ell) \prod_{\substack{k \in \bar{I} \\ k \neq j, \ell}} f_{jk} h_{\ell k} + Q(u, u_j, -u_\ell - \eta) \prod_{\substack{k \in \bar{I} \\ k \neq j, \ell}} h_{jk} f_{\ell k} \right) \end{aligned} \quad (53)$$

where  $h_{jk}$  and  $f_{jk}$  stand for  $h(u_j, u_k)$  and  $f(u_j, u_k)$  and

$$L(u, u_j) = 2 (\kappa_1(u)g(u, u_j) + \kappa_2(u)n(u, u_j)) \Xi(u_j) \frac{\bar{b}u_j - \bar{a}}{(2u_j + \eta)(u_j + \eta)} \quad (54)$$

$$Q(u, u_j, u_\ell) = -4 Z_{11}(u, u_j, u_\ell) \Xi(u_j)\Xi(u_\ell) \frac{(\bar{b}u_j - \bar{a})(\bar{b}u_\ell - \bar{a})(u_j + u_\ell + 2\eta)}{(2u_j + \eta)(2u_\ell + \eta)(u_j + u_\ell)}. \quad (55)$$

To prove relation (53), we consider its R.H.S as a function of  $u$ , called  $X(u)$  and proves that it vanishes. Firstly, it is easy to see that  $X(u)$  is a rational function that tends to 0 when  $u \rightarrow \infty$ . Secondly, we remark that  $X(u)$  can possess poles only at  $u = u_j$  and  $u = -u_j - \eta$ . Thirdly, we prove that its residues at  $u = u_j$  is equivalent to the sum over all the residues of the following function over  $z$

$$\frac{\bar{a} - \bar{b}z}{(z + u_j)(z - u_j - \eta)z} \prod_{\ell \in \bar{I}} f(z, u_\ell) \quad (56)$$

and, by consequence, vanishes. We perform the same type of computation for the poles at  $u = -u_j - \eta$ . This makes  $X(u)$  a rational function that vanishes at infinity and has no pole: it is equals to 0. This proves the projection on  $\mathcal{B}(u)|\mathbb{B}(\mathbf{u}_I)\rangle$  for  $I \in \mathcal{I}_\ell^N$  and  $\ell = 0, 1, \dots, N-2$  and concludes the proof of relation (33).

## 4 Conclusion

We have constructed the algebraic Bethe ansatz for the XXX model with open boundary conditions characterized by two general boundary matrices with one constraint. This relation amounts to state that the two boundary matrices can be triangularized in the same basis. This property corresponds to the local gauge transformations used in XXZ model to diagonalize the boundaries, and possible only when constraints are applied to the boundary matrices.

Several direction of investigations can follow: keeping the same model, one should compute the scalar products of (off-shell) Bethe vectors in order to have access to the correlation functions of the model; the same construction for the XXZ model with non-diagonal boundaries should also be done and compared with the previous works [19, 20, 21, 22]; generalization to different models such as spin chains based on algebras of higher rank or the Hubbard model should be also investigated; finally, the seek of an algebraic Bethe ansatz for general boundary matrices is a very exciting (but also very open) problem.

## A Commutation relations

Using the reflection equation (5), we can find the exchange relations between the operators  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ . For our calculations, we only need the following ones

$$[\mathcal{B}(u), \mathcal{B}(v)] = 0 \quad (57)$$

$$\begin{aligned} \mathcal{A}(u)\mathcal{B}(v) &= f(u, v)\mathcal{B}(v)\mathcal{A}(u) + g(u, v)\mathcal{B}(u)\mathcal{A}(v) + w(u, v)\mathcal{B}(u)\mathcal{D}(v), \\ \mathcal{D}(u)\mathcal{B}(v) &= h(u, v)\mathcal{B}(v)\mathcal{D}(u) + k(u, v)\mathcal{B}(u)\mathcal{D}(v) + n(u, v)\mathcal{B}(u)\mathcal{A}(v) \\ [\mathcal{C}(u), \mathcal{B}(v)] &= m(u, v)\mathcal{A}(v)\mathcal{A}(u) + l(u, v)\mathcal{A}(u)\mathcal{A}(v) + q(u, v)\mathcal{A}(v)\mathcal{D}(u) \\ &\quad + p(u, v)\mathcal{A}(u)\mathcal{D}(v) + y(u, v)\mathcal{D}(u)\mathcal{A}(v) + z(u, v)\mathcal{D}(u)\mathcal{D}(v) \end{aligned} \quad (58)$$



with

$$\begin{aligned}
f(u, v) &= \frac{(u-v-\eta)(u+v)}{(u+v+\eta)(u-v)}, & g(u, v) &= \frac{2\eta v}{(2v+\eta)(u-v)}, & w(u, v) &= \frac{-\eta}{(u+v+\eta)} \\
h(u, v) &= \frac{(u-v+\eta)(u+v+2\eta)}{(u-v)(u+v+\eta)}, & k(u, v) &= \frac{-2\eta(u+\eta)}{(u-v)(2u+\eta)}, \\
n(u, v) &= \frac{4v\eta(u+\eta)}{(u+v+\eta)(2v+\eta)(2u+\eta)}
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
m(u, v) &= \frac{2\eta u(u-v+\eta)}{(2u+\eta)(u+v+\eta)(u-v)}, & l(u, v) &= -\frac{2\eta^2 u}{(2u+\eta)(2v+\eta)(u-v)} \\
q(u, v) &= \frac{\eta(u+v)}{(u+v+\eta)(u-v)}, & p(u, v) &= -\frac{2\eta u}{(2u+\eta)(u-v)} \\
y(u, v) &= -\frac{\eta^2}{(u+v+\eta)(2v+\eta)}, & z(u, v) &= -\frac{\eta}{u+v+\eta}.
\end{aligned} \tag{60}$$

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